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Convection in a narrow annular channel rotating about its axis of symmetry

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The onset of convection in a narrow cylindrical annulus heated from below and rotating about its vertical axis of symmetry is considered in the case when the rigid cylindrical walls are thermally insulating. An analytical expression is derived for the Rayleigh number R for onset of convection as a function of rotation rate and azimuthal wavenumber. The critical value R_c for high rotation rates is much lower than the corresponding value in an extended layer. At finite amplitudes the convection flow generates a differential rotation which is antisymmetric with respect to the middle of the layer and is prograde near the outer wall for low rotation rates, but changes sign for higher values of the rotation parameter.

1. Introduction

Convection in horizontal layers of fluid heated from below and rotating about a vertical axis has long been a subject of intense research because of its applications to convection processes in the atmosphere and in the oceans. Numerous laboratory experiments have been devoted to this subject and many theoretical analyses and computer simulations have been published. For a recent introduction refer to the book by Boubnov & Golitsyn (1995). In the course of this research the importance of the sidewalls of the cylindrical or rectangular convection boxes was realized relatively late. Goldstein *et al.* (1993) were the first to demonstrate convincingly that convection modes attached to the sidewalls may set in at values of the Rayleigh number significantly below the value for onset of convection in an infinitely extended layer (Chandrasekhar 1961). A boundary layer theory for the sidewall-supported convection was developed independently by Herrmann & Busse (1993) and by Kuo & Cross (1993). Numerous experimental studies of the sidewall modes have been published; see, for instance, Liu & Ecke (1999) and references therein.

In this brief report we consider the onset of convection and its weakly nonlinear properties in the case of a narrow channel heated from below whose height is much larger than its width. As shown in figure 1 this configuration may thus be regarded as the limit opposite to that of an infinitely extended layer. In contrast to the case of a single sidewall the onset of convection is non-oscillatory. This may be expected from the property that the sidewall modes propagate with the same speed in opposite directions as long as the parallel sidewalls are sufficiently far apart. As the two sidewalls approach each other the interaction of the two sidewall modes gives rise to a steady pattern of convection. We use the term ‘annular’ for our configuration since it will be realized experimentally most easily in the narrow gap between two coaxial cylindrical walls rotating rigidly about their common axis. Since the width of

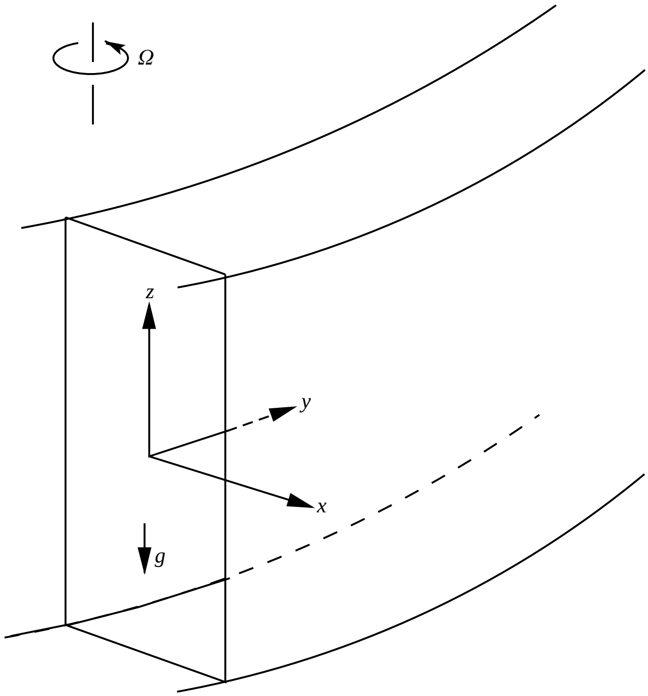

FIGURE 1. Geometrical configuration of the problem.

the annular gap is small compared to its radius it is appropriate to use a Cartesian coordinate system as indicated in figure 1 and to use periodic boundary conditions in the azimuthal y-direction.

After a short formulation of the mathematical problem in § 2 an analytical solution will be derived in § 3. Nonlinear properties such as a mean flow associated with finite-amplitude convection will be considered in § 4. Concluding remarks and outlooks on related problems will be given in § 5.

2. Mathematical formulation of the problem

We consider a fluid-filled cylindrical annulus rotating with constant angular velocity Ω about its vertical axis of symmetry. The gap width d between the two coaxial cylindrical walls is assumed to be small in comparison with their radii and their height, hd, such that locally the approximation of a straight narrow channel can be assumed as indicated in figure 1. The temperature at the horizontal bottom of the annulus is assumed to be fixed at the constant value T_2. Similarly the horizontal top boundary is kept at the lower value T_1. The vertical walls are assumed to be thermally insulating. Since effects of the centrifugal force will be neglected based on the assumption $\Omega^2 r_0 \ll g$ where r_0 is the mean radius of the annulus, a static solution of the problem with the temperature distribution $T(z) = (T_1 + T_2)/2 - (T_2 - T_1)z/h$ is possible. Here z is the dimensionless vertical coordinate measured in multiples of d.

Using d as length scale, d^2/κ as time scale where κ denotes the thermal diffusivity of the fluid, and $(T_2 - T_1)/R$ as scale of the temperature we obtain the equations of motion for the dimensionless velocity vector $\boldsymbol{v}$ and the heat equation for the deviation

Θ of the temperature from its static distribution in the following form:

$$P^{-1}(\partial_t + \boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla\pi + \boldsymbol{k}\Theta/h + \nabla^2\boldsymbol{v} - \tau h\boldsymbol{k}\times\boldsymbol{v}, \tag{2.1a}$$

$$0 = \nabla\cdot\boldsymbol{v}, \tag{2.1b}$$

$$(\partial_t + \boldsymbol{v}\cdot\nabla)\Theta = R\boldsymbol{k}\cdot\boldsymbol{v}/h + \nabla^2\Theta, \tag{2.1c}$$

where $\boldsymbol{k}$ is the unit vector in the z-direction and the Prandtl number, the Coriolis parameter, and the Rayleigh number are defined by

$$P = \frac{\nu}{\kappa}, \qquad \tau = \frac{2\Omega d^2}{\nu h}, \qquad R = \frac{\gamma(T_2 - T_1)gd^3h}{\nu\kappa}. \tag{2.2}$$

Here ν and γ denote the kinematic viscosity and the coefficient of thermal expansion of the fluid, g is the acceleration due to gravity, and h is the height to width ratio of the annular channel. The Boussinesq approximation has been assumed in which the density ρ is regarded as a constant except in connection with the gravity term. All terms in equation (2.1a) that can be written as gradients have been combined into ∇p.

It is convenient to introduce the general representation for the solenoidal velocity field $\boldsymbol{v}$,

$$\boldsymbol{v} = \nabla\times(\nabla\Phi\times\boldsymbol{i}) + \nabla\Psi\times\boldsymbol{i} \equiv \boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi, \tag{2.3}$$

where $\boldsymbol{i}$ is the unit vector in the x-direction normal to the outer wall. By taking the x-components of the (curl)^2 and of the curl of equation (2.1a) two equations for Φ and Ψ are obtained:

$$\nabla^4\Delta_2\Phi - \tau h\partial_z\Delta_2\Psi + \partial^2_{xz}\Theta/h = (\boldsymbol{\delta}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \partial_t\nabla^2\Delta_2\Phi)P^{-1}, \tag{2.4a}$$

$$\nabla^2\Delta_2\Psi + \tau h\partial_z\Delta_2\Phi - \partial_y\Theta/h = (\boldsymbol{\eta}\cdot(\boldsymbol{v}\cdot\nabla\boldsymbol{v}) + \partial_t\Delta_2\Psi)P^{-1}, \tag{2.4b}$$

where Δ_2 denotes the two-dimensional Laplacian, $\Delta_2 = \partial^2_{yy} + \partial^2_{zz}$. These equations must be solved in conjunction with the heat equation,

$$(\nabla^2 - \partial_t)\Theta + R\big(\partial^2_{xz}\Phi - \partial_y\Psi\big)/h = (\boldsymbol{\delta}\Phi + \boldsymbol{\eta}\Psi)\cdot\nabla\Theta. \tag{2.5}$$

The boundary conditions at the rigid cylindrical walls are given by

$$\Phi = \partial_x\Phi = \Psi = \partial_x\Theta = 0 \qquad \text{at} \qquad x = \pm 1/2. \tag{2.6}$$

Since we are interested in the case of large aspect ratio, $h \gg 1$, the conditions at the upper and lower boundaries are of lesser importance. Since analytical solutions can be obtained only for stress-free conditions we assume those for simplicity,

$$\partial^2_{xz}\Phi - \partial_y\Psi = \partial^3_{zzz}\Phi + \partial^3_{yyz}\Phi = \partial^3_{xyz} + \partial^2_{zz}\Psi = 0 \qquad \text{at} \quad z = \pm h/2. \tag{2.7}$$

The solution derived below obeys these conditions since it satifies

$$\partial_z\Phi = \partial^3_{zzz}\Phi = \partial_y\Psi = \partial^2_{zz}\Psi = \Theta = 0 \qquad \text{at} \quad z = \pm h/2. \tag{2.8}$$

We shall use $\epsilon \equiv \pi^2/h^2$ as expansion parameter for a perturbation analysis of the linearized version of equations (2.4) and (2.5),

$$\left.\begin{array}{ll} \Theta = \Theta_0 + \epsilon\Theta_1 + \cdots, & \Phi = \Phi_0 + \epsilon\Phi_1 + \cdots, \\ R = R_0 + \epsilon R_1 + \cdots, & \Psi = \Psi_0 + \epsilon\Psi_1 + \cdots. \end{array}\right\} \tag{2.9}$$

Anticipating that the scale of convection in the y-direction will be of the same order of magnitude as in the z-direction we write the solution of the heat equation (2.5) in lowest order:

$$\Theta_0 = \cos(\pi z/h)\exp\{\mathrm{i}\alpha\sqrt{\epsilon}y + \mathrm{i}\omega t\}, \tag{2.10}$$

where we have used the property that the boundary conditions permit an x-independent solution.

3. Solution of the linear problem

The symmetry of equations (2.4) requires that Φ_0 and Ψ_0 assume the form

$$\Phi_0 = \varphi_0(x)\sin(\pi z/h)\exp\{\mathrm{i}\alpha\sqrt{\epsilon}y + \mathrm{i}\omega t\}, \quad \Psi_0 = \psi_0(x)\cos(\pi z/h)\exp\{\mathrm{i}\alpha\sqrt{\epsilon}y + \mathrm{i}\omega t\}. \tag{3.1}$$

Anticipating that ω is a small quantity we neglect the right-hand sides of (2.4) in the linear limit of the problem and obtain through elimination of $\psi_0(x)$ the equation

$$\left(\frac{\mathrm{d}^6}{\mathrm{d}x^6} - \tau^2\pi^2\right)\varphi_0(x) = \mathrm{i}\alpha\tau/(1+\alpha^2) \tag{3.2}$$

for $\varphi_0(x)$. It yields the solution

$$\varphi_0(x) = \mathrm{i}\alpha[\tau\pi^2(1+\alpha^2)]^{-1}\hat{\varphi}(x) \quad \text{with} \quad \hat{\varphi}(x) = -1 + \sum_{n=1}^{3} a_n\cosh\lambda_n x \tag{3.3}$$

where the λ_n are given by

$$\lambda_1 = (\tau\pi)^{1/3}, \qquad \lambda_2 = \lambda_3^+ = \tfrac{1}{2}(1+\mathrm{i}\sqrt{3})(\tau\pi)^{1/3} \tag{3.4}$$

and the coefficients a_n are determined by the boundary conditions $\hat{\varphi}(x) = (\mathrm{d}/\mathrm{d}x)\hat{\varphi}(x) = (\mathrm{d}^4/\mathrm{d}x^4)\hat{\varphi}(x) = 0$ at $x = \pm 0.5$. As a result the following expressions are obtained:

$$a_1 = -\left(a_2\lambda_2\sinh\left(\lambda_2\tfrac{1}{2}\right) + a_2^+\lambda_2^+\sinh\left(\lambda_2^+\tfrac{1}{2}\right)\right)\big/\left(\lambda_1\sinh\left(\lambda_1\tfrac{1}{2}\right)\right), \tag{3.5a}$$

$$a_2 = D\big/\left[\left(\tfrac{3}{2} + \mathrm{i}\tfrac{1}{2}\sqrt{3}\right)D\cosh\left(\lambda_2\tfrac{1}{2}\right) - \text{c.c.}\right] = a_3^+, \tag{3.5b}$$

with

$$D = \tau\lambda_2^+\left(\sinh\left(\lambda_2^+\tfrac{1}{2}\right) + \tanh\left(\lambda_1\tfrac{1}{2}\right)\cosh\left(\lambda_2^+\tfrac{1}{2}\right)\right) \tag{3.6}$$

where the superscript $^+$ indicates the complex conjugate and c.c. stands for the complex conjugate of the preceding term.

At the next order the equation for Θ_1 is given by

$$\partial_{xx}^2\Theta_1 = (\mathrm{i}\omega/\epsilon + 1 + \alpha^2)\Theta_0 - R_0\left(\partial_{xz}^2\Phi_0/\sqrt{\epsilon} - \mathrm{i}\alpha\Psi_0\right)\big/\pi. \tag{3.7}$$

The solvability condition for this equation is obtained when the right-hand side is multiplied by Θ_0^+ and averaged over the fluid layer,

$$R_0\mathrm{i}\alpha\langle\overline{\Theta_0^+\Psi_0}\rangle/\pi = -(\mathrm{i}\omega/\epsilon + 1 + \alpha^2)/2 \tag{3.8}$$

where the bar indicates the average over the interval $-0.5 \leqslant x \leqslant 0.5$ and the angular brackets indicate the average over surfaces $x = \text{const.}$, i.e. over $-\infty \leqslant y \leqslant \infty$ and over $-0.5h \leqslant z \leqslant 0.5h$. ω must vanish according to this relationship because $\psi_0(x)$ is purely imaginary. Since the latter function is also proportional to $\alpha/(1+\alpha^2)$ it can already be concluded at this point that R_0 reaches its minimum as a function of α for $\alpha = 1$. Using the relationship $\psi_0(x) = -(\mathrm{d}^4\varphi_0(x)/\mathrm{d}x^4)/\tau\pi$ we can evaluate (3.8) and obtain

$$R_0 = \tau\pi^3\frac{(1+\alpha^2)^2}{2\alpha^2}\,\frac{\tanh 2\xi\,[(\sinh\xi)^2 + (\cos\hat{\xi})^2] + \sinh\xi\cosh\xi + \sqrt{3}\sin\hat{\xi}\cos\hat{\xi}}{(\sinh\xi)^2 + (\sin\hat{\xi})^2 + \tanh 2\xi(\sinh\xi\cosh\xi - \sqrt{3}\sin\hat{\xi}\cos\hat{\xi})} \tag{3.9}$$

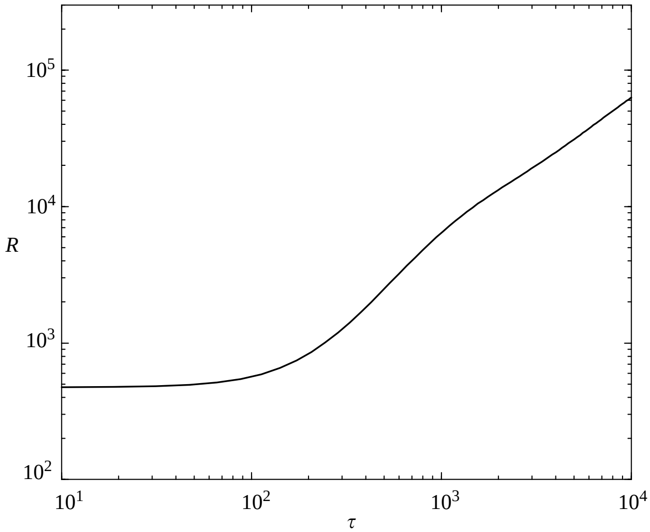

FIGURE 2. The Rayleigh number R_0 as a function of τ.

where the definitions $\xi = (\tau\pi)^{1/3}/4$ and $\hat{\xi} = \sqrt{3}\xi$ have been used. In the limit of large τ, R_0 increases in proportion to τ,

$$R_0 = \tau \frac{(1+\alpha^2)^2}{2\alpha^2}\pi^3 \quad \text{for} \quad \tau \to \infty, \tag{3.10}$$

while in the limit of vanishing τ the expression

$$R_0 = 12\pi^2 \frac{(1+\alpha^2)^2}{\alpha^2} \quad \text{for} \quad \tau = 0 \tag{3.11}$$

is obtained which corresponds to the Rayleigh number for onset of convection in a Hele-Shaw cell (Wooding 1960). A plot of the critical Rayleigh number as a function of τ is given in figure 2 which shows the smooth connection between the asymptotic relationships (3.10) and (3.11). It is remarkable that for large τ the Rayleigh number R^* based on the height of the channel, $R^* \equiv Rh^2$, depends on the rotation parameter τ^* based on the height dh, $\tau^* \equiv \tau h^3$, in such a way that convection sets in at a value of R^* which is of the order h^{-1} lower than the corresponding value for the onset of convection in an extended layer with a single sidewall, $R^* = \tau^*\pi^2(6\sqrt{3})^{1/2}$ (Herrmann & Busse 1993). This property originates from the fact that the x-dependence of Θ vanishes in first approximation and that in the presence of two parallel walls no time dependence is needed to achieve an optimal phase relationship between the variables Θ, Φ, and Ψ. The critical value of the Rayleigh number for the onset of convection in an unbounded layer is, of course, even higher for large values of τ^* since is grows proportionally to $(\tau^*)^{4/3}$ according to Chandrasekhar (1961).

The functions $\hat{\varphi}(x)$ and $\hat{\psi}(x) \equiv (\mathrm{d}^4/\mathrm{d}x^4)\hat{\varphi}(x)/(\pi\tau)^{4/3}$ are shown in figure 3. For $\tau = 10$ the functions differ little from their counterparts in the non-rotating case. In particular, $\hat{\psi}(x)$ approaches the parabolic profile assumed in the Hele-Shaw cell. Computations of convection in non-rotating channels have been done by Davies-Jones (1970). His results demonstrate quite well that the x-independence of Θ is closely approached even though his maximum value of h is only 2. At high values of τ the boundary layer character of the velocity field becomes apparent, with $\hat{\psi}(x)$

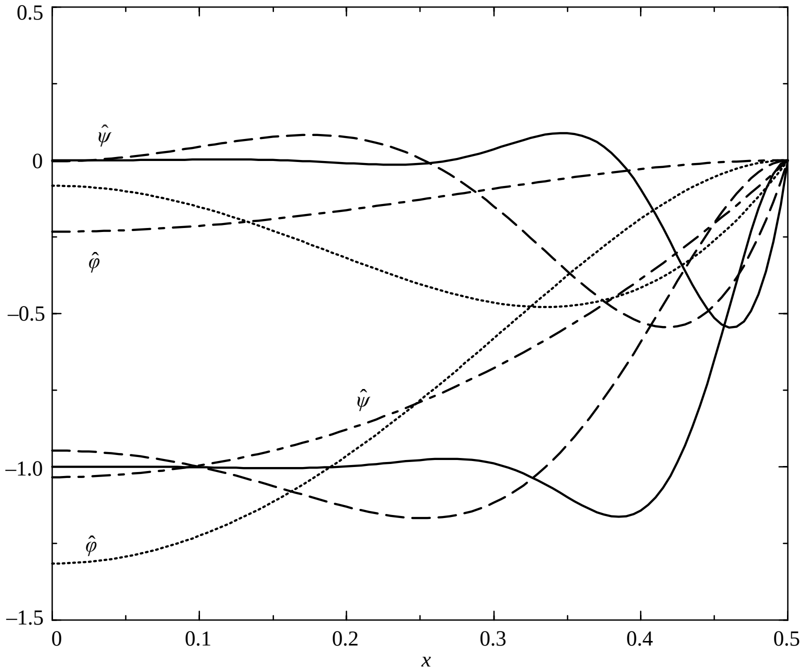

FIGURE 3. The functions $\hat{\varphi}(x)$ and $\hat{\psi}(x) \equiv (\mathrm{d}^4/\mathrm{d}x^4)\hat{\varphi}(x)/(\pi\tau)^{4/3}$ are shown for $\tau = 10$ (dash-dotted line), $\tau = 100$ (dotted line), $\tau = 1000$ (dashed line), and $\tau = 10000$ (solid line). Since the functions are symmetric in x they have been plotted only for positive x. Except for the case $\tau = 10$, $\hat{\varphi}(x)$ is close to -1 and $\hat{\psi}(x)$ is close to 0 at $x = 0$.

vanishing in the interior while $\hat{\varphi}(x)$ approaches unity there as v_x satisfies the thermal wind balance, $\tau h \partial_z v_x = -\partial_y \Theta / h$. The typical horizontal scale of the wall-attached convection scales with $\tau^{-1/3}$ and its form approaches that shown, for instance, in the boundary layer analysis of Herrmann & Busse (1993). A typical feature, for instance, is the change of sign of $\hat{\psi}(x)$ at the distance of $2\tau^{-1/3}$ from the walls. The $\tau^{-1/3}$-scale at the sidewall corresponds to the familiar $E^{1/3}$-scaling of Stewartson layers at walls parallel to the axis of rotation where the Ekman number $E = 2/\tau^*$ is defined with the height of the layer.

In principle axisymmetric forms of convection are also possible corresponding to y-independent solutions of (2.1). But since the Rayleigh number for the onset of these solutions will be of the order ϵ^{-1}, there is no need to consider them here.

4. Nonlinear properties

In order to attack the weakly nonlinear problem we extend the representation (2.8) by considering the double expansion

$$\left.\begin{aligned} \Theta &= \sum_{m=1,n=0}^{\infty} A^m \epsilon^n \Theta_{mn}, \qquad & \Phi &= \sum_{m=1,n=0}^{\infty} A^m \epsilon^n \Phi_{mn}, \\ \Psi &= \sum_{m=1,n=0}^{\infty} A^m \epsilon^n \Psi_{mn}, \qquad & R &= \sum_{m=0,n=0}^{\infty} A^m \epsilon^n R_{mn}, \end{aligned}\right\} \tag{4.1}$$

where A measures the amplitude of convection and where Θ_{10}, Φ_{10}, Ψ_{10} correspond to the real parts of Θ_0, Φ_0, Ψ_0. R_{00} is identical to R_0, of course. We define the amplitude A by $A = 4\langle\overline{\Theta\Theta_{10}}\rangle$ such that the normalization condition

$$\langle\overline{\Theta_{mn}\Theta_{10}}\rangle = \delta_{0n}\delta_{1m}/4 \tag{4.2}$$

is obtained. A main task is the determination of the coefficients R_{10}, R_{20} which describe the dependence of the amplitude A on the supercritical Rayleigh number. We shall focus attention first on the limit of large Prandtl numbers in which case the terms on the right-hand sides of (2.4) can be neglected.

The equation $\partial^2_{xx}\Theta_{20} = 0$ at order $A^2\epsilon^0$ is solved by $\Theta_{20} = g(y, z)$ where $g(y, z)$ must still be determined subject to the boundary conditions and the normalization condition (4.2). In the order $A^2\epsilon^1$ we obtain the equation

$$\begin{aligned} &\partial^2_{xx}\Theta_{21} + R_{00}\left(\partial^2_{xz}\Phi_{20} - \partial_y\Psi_{20}\right)\big/\sqrt{\epsilon}\pi \\ &= \left(\partial^2_{yy} + \partial^2_{zz}\right)\Theta_{20}/\epsilon - R_{10}\left(\partial^2_{xz}\Phi_{10} - \partial_y\Psi_{10}\right)\big/\sqrt{\epsilon}\pi + (\boldsymbol{\delta}\Phi_{10} + \boldsymbol{\eta}\Psi_{10})\cdot\nabla\Theta_{10}/\epsilon. \end{aligned} \tag{4.3}$$

The factors $\sqrt{\epsilon}$ and ϵ appear in this equation since they are needed to compensate the corresponding derivatives in order that all terms are of order unity. The solvability condition for equation (4.3) is obtained when the right-hand side is multiplied by Θ_{10} and averaged over the fluid layer as indicated by the overline together with the angular brackets,

$$\begin{aligned} -\frac{\sqrt{\epsilon}}{\pi}R_{10}\langle\overline{\Theta_{10}\partial_y\Psi_{10}}\rangle &= \langle\overline{\Theta_{10}(\boldsymbol{\delta}\Phi_{10} + \boldsymbol{\eta}\Psi_{10})\cdot\nabla\Theta_{10}}\rangle \\ &= \langle\overline{\nabla\cdot((\boldsymbol{\delta}\Phi_{10} + \boldsymbol{\eta}\Psi_{10})(\Theta_{10})^2/2)}\rangle = 0 \end{aligned} \tag{4.4}$$

Here we have used that the terms involving Θ_{20} and Ψ_{20} in equation (4.3) do not contribute in the solvability condition (4.4) because of the normalization condition (4.2). Since R_{10} vanishes according to equation (4.4) we obtain the following equation for Θ_{20} by averaging equation (4.3) over x as is indicated by the overline:

$$\left(\partial^2_{yy} + \partial^2_{zz}\right)\Theta_{20} = \overline{\boldsymbol{\eta}\Psi_{10}\cdot\nabla\Theta_{10}} = -\frac{\epsilon\alpha^2\overline{\mathrm{d}^4\hat{\varphi}(x)/\mathrm{d}x^4}}{2\tau^2\pi^3(1+\alpha^2)}\sin(2\pi z/h). \tag{4.5}$$

Because the y-dependence of Θ_{20} has vanished, Φ_{20} and Ψ_{20} vanish as well.

In order to determine R_{20} the solvability condition for the equation for Θ_{31} must be considered, which assumes the form

$$\begin{aligned} -\frac{\sqrt{\epsilon}}{\pi}R_{20}\langle\overline{\Theta_{10}\partial_y\Psi_{10}}\rangle &= \langle\overline{\Theta_{10}(\boldsymbol{\delta}\Phi_{10} + \boldsymbol{\eta}\Psi_{10})\cdot\nabla\Theta_{20}}\rangle \\ &= -\langle\overline{\partial_y\Psi_{10}\Theta_{10}\partial_z\Theta_{20}}\rangle = \epsilon\left(\frac{\alpha^2\overline{\mathrm{d}^4\hat{\varphi}(x)/\mathrm{d}x^4}}{4\tau^2\pi^3(1+\alpha^2)}\right)^2\Bigg/2 \end{aligned} \tag{4.6}$$

in analogy to condition (4.4). Using equation (3.3) we obtain an expression for R_{20} in the form

$$R_{20} = \frac{(1+\alpha^2)\pi^2}{8R_0}. \tag{4.7}$$

A detailed inspection of the right-hand sides of equations (2.4) reveals that those terms cannot contribute to R_{20} and will become relevant only at higher orders of ϵ. The expression (4.7) is thus valid for arbitrary Prandtl numbers and the restriction mentioned above is thus not necessary. An important nonlinear property of convection is the Nusselt number Nu which can now be derived as a function of $R-R_0$,

$$Nu-1=-hA^2\partial_z\Theta_{20}|_{z=0.5}/R=-\frac{\alpha^2\overline{\mathrm{d}^4\hat{\varphi}(x)/\mathrm{d}x^4}(R-R_0)}{4\tau^2\pi^2(1+\alpha^2)RR_{20}}=\frac{2(R-R_0)}{R}. \tag{4.8}$$

It is remarkable that the expression on the right-hand side does not depend explicitly on the rotation parameter τ. The influence of rotation enter only through R_0. In the limit $\tau=0$ the expression for the Nusselt number is the same as that which has been derived for convection rolls in a horizontal fluid-filled porous layer (see, for instance, Palm, Weber & Kvernvold 1972 or Chapter XI of Joseph 1976). The latter problem is, of course, mathematically identical to the problem of convection in a Hele-Shaw cell.

Among the nonlinear properties induced by convection the mean flow in the azimuthal direction, $\langle\partial_z\Psi\rangle$, is of special interest. It is determined by the average over surfaces $x=$const. of equation (2.1a),

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\langle\partial_z\Psi\rangle=-\frac{\mathrm{d}}{\mathrm{d}x}\left\langle\Delta_2\Phi\left(\partial^2_{xy}\Phi+\partial_z\Psi\right)\right\rangle P^{-1}. \tag{4.9}$$

At the lowest orders of ϵ $\langle\partial_z\Psi\rangle$ vanishes, $\langle\partial_z\Psi_{m0}\rangle=0$ for $m\geqslant1$. But for $\langle\partial_z\Psi_{21}\rangle$ a finite result is obtained,

$$\begin{aligned}\epsilon\frac{\mathrm{d}^2}{\mathrm{d}x^2}\langle\partial_z\Psi_{21}\rangle&=-\frac{\mathrm{d}}{\mathrm{d}x}\left\langle\Delta_2\Phi_{10}\left(\partial^2_{xy}\Phi_{10}+\partial_z\Psi_{10}\right)\right\rangle P^{-1}\\&=\frac{\mathrm{d}}{\mathrm{d}x}\left(\hat{\varphi}(x)\frac{\mathrm{d}^4\hat{\varphi}(x)}{\mathrm{d}x^4}\right)\frac{\alpha^2\epsilon}{4\tau^3(1+\alpha^2)\pi^4hP}.\end{aligned} \tag{4.10}$$

At lowest order the mean flow $U_y(x)$ is thus given by

$$U_y(x)=\epsilon A^2\langle\partial_z\Psi_{21}\rangle=A^2\left(\frac{\pi}{h}\right)^3\frac{\alpha^2}{4(\tau\pi)^{5/3}(1+\alpha^2)P}\hat{u}(x) \tag{4.11}$$

where $\hat{u}(x)$ is defined by

$$\hat{u}(x)=\left(\int_0^x\hat{\varphi}(\hat{x})\frac{\mathrm{d}^4\hat{\varphi}(\hat{x})}{\mathrm{d}\hat{x}^4}\mathrm{d}\hat{x}-2x\int_0^{1/2}\hat{\varphi}(\hat{x})\frac{\mathrm{d}^4\hat{\varphi}(\hat{x})}{\mathrm{d}\hat{x}^4}\mathrm{d}\hat{x}\right)\Big/(\tau\pi)^{4/3} \tag{4.12}$$

where the property has been used that the integrand of the integrals is symmetric with respect to $x=0$. Evidently expression (4.12) satisfies the boundary conditions $\hat{u}=0$ at $x=\pm\frac{1}{2}$. An explicit analytical expression for U_y could be derived, but because of its complexity it will not be given here.

5. Concluding remarks

The problem treated in this paper for which gravity acts parallel to the axis of rotation must be clearly distinguished from the case of convection in a cylindrical annulus with gravity acting at a right-angle to the axis of rotation (Busse 1970). The latter problem is usually realized through the use of the centrifugal force as effective gravity. The analysis of the present paper neglects the centrifugal force and is thus limited by the condition $\Omega^2r_0\ll g$ where r_0 is the mean radius of the cylindrical annulus. Nevertheless high values of τ may be realized if sufficiently low values of the kinematic viscosity are used. Inclusion of the centrifugal force would prevent the static basic solution on which the analysis has been based.

The symmetry of the solutions that we have considered is such that higher-order contributions will not change the symmetry of the mean flow U_y. For this reason

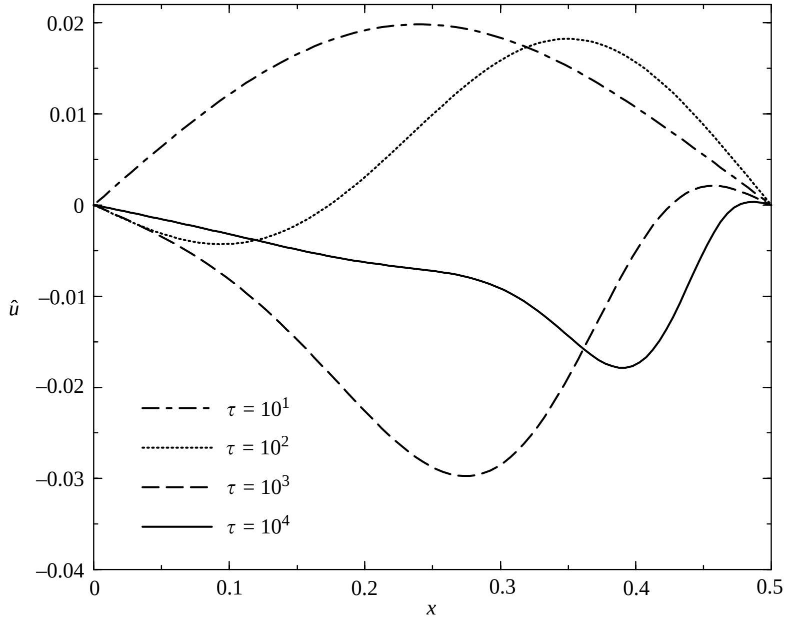

FIGURE 4. The mean flow profile $\hat{u}(x)$ for the values of τ as indicated.

there will not be an advection of the convection pattern by the mean flow and the frequency ω will vanish in all orders of the problem. In a laboratory experiment, of course, deviations from the asymptotic small gap limit cannot be avoided in general and small asymmetries with respect to the mid-surface of the annular channel must be expected. Slow drifts of the convection pattern are thus likely to be observed in experimental realizations of the problem. The mean flow induced by convection is an especially interesting feature of the problem. It has already been considered in the early work of Davies-Jones & Gilman (1971) where, by the way, convection modes propagating along the rigid sidewalls were computed for the first time. For their steady convection mode these authors find for $\tau = 80$ an antisymmetric differential rotation with prograde motion on the outside in general agreement with the results displayed in figure 4. Quantitative comparisons with their computational results are not possible since they used only the value $h = 0.5$. Another computation of a mean flow induced by sidewall-attached convection in a rotating annular channel has been carried out by Plaut (2003). For values of τ of the order 200 he finds that the mean flow is mainly retrograde in agreement with the results shown in figure 4. It thus appears that the changeover from a prograde mean flow on the outside to a retrograde one with increasing τ as seen in figure 4 is a fairly robust feature of convection in rotating annular channels.

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